PASCAL'S TRIANGLE & THE BINOMIAL THEOREM

You have multiplied binomials using FOIL and the distributive property, but when a binomial is raised to a power greater than two the problem gets more difficult and more ugly. Thankfully mathematicians have Pascal's Triangle and the Binomial Theorem.

Let's expand some binomials and see what happens. To expand, first multiply, and then write the result as a polynomial in standard form.

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
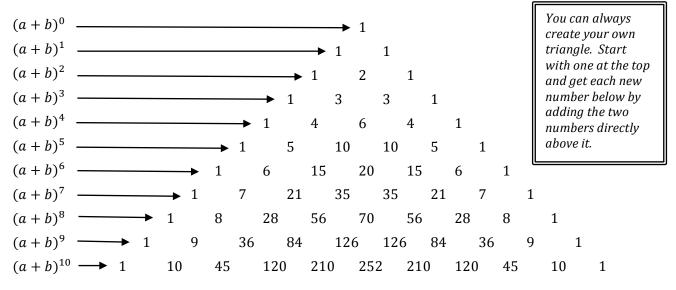$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$
$$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

If you arrange the coefficients of each polynomial in a triangle you get **Pascal's Triangle**.

HISTORY

The earliest known version of Pascal's Triangle was developed between 300 and 200 BC by the Indian mathematician Halayudha. It also appeared in a Chinese publication in AD 1303 and Persian mathematician Omar Khayyam had knowledge of it around AD 1100. It is, however, named for mathematician Blaise Pascale who lived 1623-1662.

DEF: **Pascal's Triangle** is a triangular arrangment of numbers. It is created by writing ones along the two sides and writing the sum of adjacent numbers below. It can be used to calculate polynomial coefficients during binomial expansion.
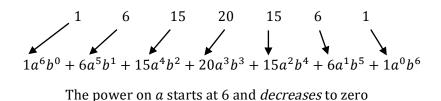
$(a + b)^0$					1						
$(a + b)^1$				1		1					
$(a + b)^2$				1	2	1					
$(a + b)^3$			1	3		3	1				
$(a + b)^4$		1	4	6		4	1				
$(a + b)^5$		1	5	10		10	5	1			
$(a + b)^6$	1	6	15	20		15	6	1			
$(a + b)^7$	1	7	21	35		35	21	7	1		
$(a + b)^8$	1	8	28	56		70	56	28	8	1	
$(a + b)^9$	1	9	36	84	126	126	84	36	9	1	
$(a + b)^{10}$	1	10	45	120	210	252	210	120	45	10	1

You can always create your own triangle. Start with one at the top and get each new number below by adding the two numbers directly above it.

Each row of Pascal's Triangle contains coefficients for the expansion $(a + b)^n$. For example, when $n = 6$ you can find the coefficients for the expansion $(a + b)^6$ in the row that has 6 as the second number.

RULE   To expand a binomial $(a + b)^n$ using Pascal's Triangle:

1. Create a Pascal's Triangle and find the correct row. The exponent n on the binomial will be equal to the second number in the row you need.
2. Make each of those numbers the coefficient on a term ab and separate these terms with addition.
3. Put exponents on the a and b of each term. The exponents on a will start at n and decrease by one until they reach zero. The exponents on b will start at zero and increase by one until they reach n. The total exponents on each term will add to n.
4. Simplify.

EXAMPLE   Use Pascal's Triangle to expand $(a + b)^6$.

Each number in the triangle becomes a coefficient of a term in the polynomial. Looking at the triangle, find the row with 6 as the second number. Take these numbers and attach them to ab terms with the exponents on a decreasing from left to right and the exponents on b increasing. Notice how the exponents of each term will add to 6, which is n.

$$1 \quad\quad 6 \quad\quad 15 \quad\quad 20 \quad\quad 15 \quad\quad 6 \quad\quad 1$$

$$1a^6b^0 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1a^0b^6$$

The power on a starts at 6 and *decreases* to zero

The power on b starts at 0 and *increases* to 6

Now simplify, remembering that anything raised to the zero power is equal to one.

$$a^6 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + b^6$$

PRACTICE   Use Pascal's Triangle to expand
1. $(a + b)^8$
2. $(a + b)^{10}$

Wow! That isn't too bad. But what if you need to expand a binomial other than $(a + b)$? What about something like $(x - 2)^3$? You can follow the exact same steps except after writing the expansion substitute $a = x$ and $b = -2$. Then simplify using your knowledge of exponents and negative numbers.

Use Pascal's Triangle to expand $(x - 2)^3$.

Follow the same steps as in Example 1. First expand $(a + b)^3$:

$$1 \quad\quad 3 \quad\quad 3 \quad\quad 1$$
$$1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

Then let $a = x$ and $b = -2$. Substitute into the polynomial and simplify.

$$1x^3(-2)^0 + 3x^2(-2)^1 + 3x^1(-2)^2 + 1x^0(-2)^3$$
$$1x^3(1) + 3x^2(-2) + 3x^1(4) + 1x^0(-8)$$
$$x^3 - 6x^2 + 12x - 8$$

> **Remember!**
> $2^3 = 2 \cdot 2 \cdot 2 = 8$

Use Pascal's Triangle to expand each binomial.

1. $(x - 2)^4$ 2. $(x - 3)^3$ 3. $(x - 4)^8$

But what if you need to expand a binomial and n is something very large like 100? Or what if you only need to know about one term in the middle of the polynomial? The relationship in Pascal's Triangle has been summarized in the **Binomial Theorem**. It gives you a short cut around Pascal's Triangle when dealing with large powers.

THE BINOMIAL THEOREM

For every positive integer n:

$$(a + b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_{n-1}ab^{n-1} + {}_nC_na^0b^n$$

Or written without combinations:

$$(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots + 1a^0b^n$$

A selection in which order does *not* matter is a **combination.** The Binomial Theorem uses combinations to describe the pattern for the coefficients in a binomial expansion. Remember:

$$_nC_r = \frac{n!}{r!\,(n-r)!} \quad for\ 0 \le r \le n$$

$n!$ is **factorial** notation and means multiply n by every number under it all the way down to 1. For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. By definition, $0! = 1$.

To expand a binomial using the Binomial Theorem:

1. Substitute the value of n into the Binomial Theorem to get the pattern for the terms.
2. Solve each combination using the formula for combinations. These numbers become the coefficients of each term.
3. Simplify.
4. If necessary, substitute for a and b and simplify to arrive at the final expansion.

EXAMPLE   Use the Binomial Theorem to expand $(a + b)^4$.

Substitute $n = 4$ into the Binomial Theorem then simplify and evaluate each combination.

$$(a + b)^4 = {}_4C_0 a^4 b^0 + {}_4C_1 a^{4-1} b^1 + {}_4C_2 a^{4-2} b^2 + {}_4C_3 a^{4-3} b^3 + {}_4C_4 a^{4-4} b^4$$
$$= {}_4C_0 a^4 + {}_4C_1 a^3 b^1 + {}_4C_2 a^2 b^2 + {}_4C_3 a^1 b^3 + {}_4C_4 b^4$$

$$_4C_0 = \frac{4!}{0!\,(4-0)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(1) \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{24}{24} = 1$$
$$_4C_1 = \frac{4!}{1!\,(4-1)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(1) \cdot 3 \cdot 2 \cdot 1} = \frac{24}{6} = 4$$
$$_4C_2 = \frac{4!}{2!\,(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot 2 \cdot 1} = \frac{24}{4} = 6$$
$$_4C_3 = \frac{4!}{3!\,(4-3!)} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \cdot 1} = \frac{24}{6} = 4$$
$$_4C_4 = \frac{4!}{4!\,(4-4)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(1)} = \frac{24}{24} = 1$$

Look back at the Pascal's Triangle. Notice these are the same numbers found in the fifth row of the triangle! Substitute the value of each combination back into the equation to get the final expansion:

$$(a + b)^4 = 1a^4 + 4a^3 b^1 + 6a^2 b^2 + 4a^1 b^3 + 1b^4$$
$$= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$

If you were expanding a binomial like $(x - 5)^4$ you would now substitute $a = x$ and $b = -5$ and do the final simplification. It would look like this:

$$(a + b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$
$$(x - 5)^4 = x^4 + 4x^3(-5) + 6x^2(-5)^2 + 4x(-5)^3 + (-5)^4$$
$$(x - 5)^4 = x^4 - 20x^3 + 6x^2(25) + 4x(-125) + 625$$
$$(x - 5)^4 = x^4 - 20x^3 + 150x^2 - 500x + 625$$